

INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

DEPARTMENT OF MATHEMATICS

PROBLEM SHEET 2

MA 4010 : Real Analysis

The Euclidean metric on $\mathbb{R}^n$ is given by $d(\bar{x}, \bar{y}) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$ for $\bar{x} \in \mathbb{R}^n, \bar{y} \in \mathbb{R}^n$. Through out, let X, Y denote metric spaces. **Answer all parts of the same question at the same place.**

1. For any two real sequences $\{a_n\}, \{b_n\}$, if $\sum a_n$ converges, and if b_n is a monotonic and bounded, prove that $\sum a_n b_n$ converges.
2. Suppose $\{p_n\}$ and $\{q_n\}$ are two Cauchy sequences in a metric space X . Show that $\{d(p_n, q_n)\}$ converges.
3. Suppose $f : (a, b) \rightarrow \mathbb{R}$ is a continuous function. Can you always extend this function to $[a, b]$ continuously? Answer the same, if f is uniformly continuous.
4. If E_n is a sequence of closed and bounded sets in a complete metric space X , if $E_n \supseteq E_{n+1}$ and if $\lim_{n \rightarrow \infty} \text{diam}(E_n) = 0$, then $\bigcap_{n=1}^{\infty} E_n$ consist of exactly one point.
5. Investigate the convergence or divergence of $\sum a_n$, if
 - (a) $a_n = \sqrt{n+1} - \sqrt{n}$.
 - (b) $a_n = (\sqrt[n]{n} - 1)^n$.
 - (c) $a_n = \frac{1}{\log n}$.
6. Find the radius of the convergence of the following power series:
 - (a) $\sum n^3 z^n$.
 - (b) $\sum \frac{n^3}{3^n} z^n$.
7. If f is a real valued continuous function defined on a closed set $E \subseteq \mathbb{R}$, prove that there exist a continuous real function g on $\mathbb{R}$ such that $f = g$ on E . Show that result become false if the word 'closed' omitted.
8. Suppose $A \subseteq \mathbb{R}$ and $f, g : A \rightarrow \mathbb{R}$ are uniformly continuous functions. Prove or disprove that fg is uniformly continuous.